Minimal Affinizations of Representations of Quantum Groups: the $U_q(\mathfrak{g})$–module structure

Vyjayanthi Chari[1],

Department of Mathematics,

University of California, Riverside, CA 92521, USA.

Abstract

If $U_q(\mathfrak{g})$ is a finite–dimensional complex simple Lie algebra, an affinization of a finite–dimensional irreducible representation V of $U_q(\mathfrak{g})$ is a finite–dimensional irreducible representation $\hat{V}$ of $U_q(\hat{\mathfrak{g}})$ which contains V with multiplicity one, and is such that all other $U_q(\mathfrak{g})$–types in $\hat{V}$ have highest weights strictly smaller than that of V. There is a natural partial ordering $\preceq$ on the set of affinizations of V defined in [2]. If $\mathfrak{g}$ is of rank 2, we prove in [2] that there is unique minimal element with repsect to this order. In this paper, we give the $U_q(\mathfrak{g})$–module structure of the minimal affinization when $\mathfrak{g}$ is of type B_2.

Introduction

In [2], we defined the notion of an affinization of a finite-dimensional irreducible representation V of the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a finite-dimensional complex simple Lie algebra and $q \in \mathbb{C}^\times$ is transcendental. An affinization of V is an irreducible representation $\hat{V}$ of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ which, regarded as a representation of $U_q(\mathfrak{g})$, contains V with multiplicity one, and is such that all other irreducible components of $\hat{V}$ have highest weights strictly smaller than that of V. We say that two affinizations are equivalent if they are isomorphic as representations of $U_q(\mathfrak{g})$. We refer the reader to the introduction to [2] for a discussion of the significance of the notion of an affinization.

An interesting problem is to describe the structure of $\hat{V}$ as a representation of $U_q(\mathfrak{g})$. This problem appears difficult for an arbitrary affinization; however, in [2] we introduced a partial order on the set of equivalence classes of affinizations of V and proved that there is a unique minimal affinization if $\mathfrak{g}$ is of rank 2. If $\mathfrak{g}$ is of type A, it was known that every V has an affinization $\hat{V}$ which is irreducible under $U_q(\mathfrak{g})$; it was proved in [4] that $\hat{V}$ is the unique minimal affinization up to equivalence. However, if $\mathfrak{g}$ is not of type A, there is generally no affinization of

[1]Partially supported by the NSF, DMS-9207701.

a given representation V which is irreducible under $U_q(\mathfrak{g})$ and the description of the structure of the minimal affinizations as representations of $U_q(\mathfrak{g})$ is not obvious. Some examples were worked out in [7]; in this paper, we describe the $U_q(\mathfrak{g})$-structure of the minimal affinization of an arbitrary irreducible representation of V when $\mathfrak{g}$ is of type B_2. A consequence of our results is that the minimal affinizaton of V is irreducible under $U_q(\mathfrak{g})$ if and only if the value of the highest weight on the short simple root of $\mathfrak{g}$ is 0 or 1.

1 Quantum affine algebras and their representations

In this section, we collect the results about quantum affine algebras which we shall need later.

Let $\mathfrak{g}$ be a finite–dimensional complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Cartan matrix $A = (a_{ij})_{i,j\in I}$. Fix coprime positive integers $(d_i)_{i\in I}$ such that $(d_i a_{ij})$ is symmetric. Let $P = \mathbb{Z}^I$ and let $P^+ = \{\lambda \in P \mid \lambda(i) \geq 0 \text{ for all } i \in I\}$. Let R (resp. R^+) be the set of roots (resp. positive roots) of $\mathfrak{g}$. Let α_i ($i \in I$) be the simple roots and let θ be the highest root. Define a non-degenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = d_i a_{ij}$, and set $d_0 = \frac{1}{2}(\theta, \theta)$. Let $Q = \oplus_{i\in I}\mathbb{Z}.\alpha_i \subset \mathfrak{h}^*$ be the root lattice, and set $Q^+ = \sum_{i\in I}\mathbb{N}.\alpha_i$. Define a partial order $\geq$ on P by $\lambda \geq \mu$ iff $\lambda - \mu \in Q^+$. Let λ_i (($i \in I$)) be the fundamental weights of $\mathfrak{g}$, so that $\lambda_i(j) = \delta_{ij}$.

In this paper, we shall be interested in the case when $\mathfrak{g}$ is of type B_2. Then,

$$I = \{1, 2\}, \qquad d_0 = d_1 = 2, \quad d_2 = 1, \quad \theta = \alpha_1 + 2\alpha_2,$$
$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

Let $q \in \mathbb{C}^\times$ be transcendental, and, for $r, n \in \mathbb{N}$, $n \geq r$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$
$$[n]_q! = [n]_q[n-1]_q \ldots [2]_q[1]_q,$$
$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}.$$

Proposition 1.1. *There is a Hopf algebra $U_q(\mathfrak{g})$ over $\mathbb{C}$ which is generated as an algebra by elements $x_i^\pm$, $k_i^{\pm 1}$ ($i \in I$), with the following defining relations:*

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,$$
$$k_i x_j^\pm k_i^{-1} = q_i^{\pm a_{ij}} x_j^\pm,$$
$$[x_i^+, x_j^-] = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum^{1-a_{ij}} \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix} (x_i^\pm)^r x_j^\pm (x_i^\pm)^{1-a_{ij}-r} = 0, \qquad i \neq j.$$

The comultiplication Δ, counit ϵ, and antipode S of $U_q(\mathfrak{g})$ are given by

$$\Delta(x_i^+) = x_i^+ \otimes k_i + 1 \otimes x_i^+,$$
$$\Delta(x_i^-) = x_i^- \otimes 1 + k_i^{-1} \otimes x_i^-,$$
$$\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1},$$
$$\epsilon(x_i^\pm) = 0, \ \ \epsilon(k_i^{\pm 1}) = 1,$$
$$S(x_i^+) = -x_i^+ k_i^{-1}, \ S(x_i^-) = -k_i x_i^-, \ S(k_i^{\pm 1}) = k_i^{\mp 1},$$

for all $i \in I$. $\square$

Let $\hat{I} = I \amalg \{0\}$ and let $\hat{A} = (a_{ij})_{i,j \in \hat{I}}$ be the extended Cartan matrix of $\mathfrak{g}$, i.e. the generalized Cartan matrix of the (untwisted) affine Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$. Let $q_0 = q^{d_0}$.

When $\mathfrak{g}$ is of type B_2,

$$\hat{A} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix}.$$

Theorem 1.2. *Let $U_q(\hat{\mathfrak{g}})$ be the algebra with generators $x_i^\pm$, $k_i^{\pm 1}$ ($i \in \hat{I}$) and defining relations those in 1.1, but with the indices i, j allowed to be arbitrary elements of $\hat{I}$. Then, $U_q(\hat{\mathfrak{g}})$ is a Hopf algebra with comultiplication, counit and antipode given by the same formulas as in 1.1 (but with $i \in \hat{I}$).*

Moreover, $U_q(\hat{\mathfrak{g}})$ is isomorphic to the algebra $\mathcal{A}_q$ with generators $x_{i,r}^\pm$ ($i \in I$, $r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I$, $r \in \mathbb{Z} \backslash \{0\}$) and $c^{\pm 1/2}$, and the following defining relations:

$$c^{\pm 1/2} \ \textit{are central},$$
$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \ \ c^{1/2} c^{-1/2} = c^{-1/2} c^{1/2} = 1,$$
$$k_i k_j = k_j k_i, \ \ k_i h_{j,r} = h_{j,r} k_i,$$
$$k_i x_{j,r} k_i^{-1} = q_i^{\pm a_{ij}} x_{j,r}^\pm,$$
$$[h_{i,r}, x_{j,s}^\pm] = \pm \frac{1}{r} [r a_{ij}]_{q_i} c^{\mp |r|/2} x_{j,r+s}^\pm,$$
$$x_{i,r+1}^\pm x_{j,s}^\pm - q_i^{\pm a_{ij}} x_{j,s}^\pm x_{i,r+1}^\pm = q_i^{\pm a_{ij}} x_{i,r}^\pm x_{j,s+1}^\pm - x_{j,s+1}^\pm x_{i,r}^\pm, \tag{1}$$
$$[h_{i,r}, h_{j,s}] = \delta_{r,-s} \frac{1}{r} [r a_{ij}]_{q_i} \frac{c^r - c^{-r}}{q_j - q_j^{-1}},$$
$$[x_{i,r}^+, x_{j,s}^-] = \delta_{ij} \frac{c^{(r-s)/2} \phi_{i,r+s}^+ - c^{-(r-s)/2} \phi_{i,r+s}^-}{q_i - q_i^{-1}},$$

$$\sum_{\pi \in \Sigma_m} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} x_{i,r_{\pi(1)}}^\pm \ldots x_{i,r_{\pi(k)}}^\pm x_{j,s}^\pm x_{i,r_{\pi(k+1)}}^\pm \ldots x_{i,r_{\pi(m)}}^\pm = 0, \tag{2}$$

if $i \neq j$, for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}$, Σ_m is the symmetric group on m letters, and the $\phi_{i,r}^\pm$ are determined by equating powers of u in the formal power series

$$\sum_{r=0}^{\infty} \phi_{i,\pm r}^\pm u^{\pm r} = k_i^{\pm 1} exp \left(\pm (q_i - q_i^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^{\pm s} \right).$$

If $\theta = \sum_{i\in I} m_i\alpha_i$, *set* $k_\theta = \prod_{i\in I} k_i^{m_i}$. *Suppose that the root vector* $\overline{x}_\theta^+$ *of* $\mathfrak{g}$ *corresponding to* θ *is expressed in terms of the simple root vectors* $\overline{x}_i^+$ $(i \in I)$ *of* $\mathfrak{g}$ *as*

$$\overline{x}_\theta^+ = \lambda[\overline{x}_{i_1}^+, [\overline{x}_{i_2}^+, \cdots, [\overline{x}_{i_k}^+, \overline{x}_j^+]\cdots]]$$

for some $\lambda \in \mathbb{C}^\times$. *Define maps* $w_i^\pm : U_q(\hat{\mathfrak{g}}) \to U_q(\hat{\mathfrak{g}})$ *by*

$$w_i^\pm(a) = x_{i,0}^\pm a - k_i^{\pm 1} a k_i^{\mp 1} x_{i,0}^\pm.$$

Then, the isomorphism $f : U_q(\hat{\mathfrak{g}}) \to \mathcal{A}_q$ *is defined on generators by*

$$f(k_0) = k_\theta^{-1},\ f(k_i) = k_i,\ f(x_i^\pm) = x_{i,0}^\pm, \qquad (i \in I),$$
$$f(x_0^+) = \mu w_{i_1}^- \cdots w_{i_k}^-(x_{j,1}^-)k_\theta^{-1},$$
$$f(x_0^-) = \lambda k_\theta w_{i_1}^+ \cdots w_{i_k}^+(x_{j,-1}^+),$$

where $\mu \in \mathbb{C}^\times$ *is determined by the condition*

$$[x_0^+, x_0^-] = \frac{k_0 - k_0^{-1}}{q_0 - q_0^{-1}}. \quad \square$$

See [1], [5] and [9] for further details.

Note that there is a canonical homomorphism $U_q(\mathfrak{g}) \to U_q(\hat{\mathfrak{g}})$ such that $x_i^\pm \mapsto x_i^\pm$, $k_i^{\pm 1} \mapsto k_i^{\pm 1}$ for all $i \in I$. Thus, any representation of $U_q(\hat{\mathfrak{g}})$ may be regarded as a representation of $U_q(\mathfrak{g})$.

It is easy to see that, for any $a \in \mathbb{C}^\times$, there is a Hopf algebra automorphism τ_a of $U_q(\hat{\mathfrak{g}})$ given by

$$\tau_a(x_{i,r}^\pm) = a^r x_{i,r}^\pm,\ \tau_a(\phi_{i,r}^\pm) = a^r \phi_{i,r}^\pm,$$
$$\tau_a(c^{\frac{1}{2}}) = c^{\frac{1}{2}},\ \tau_a(k_i) = k_i,$$

for $i \in I$, $r \in \mathbb{Z}$ (see [5]).

Let $\hat{U}^\pm$ (resp. $\hat{U}^0$) be the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by the $x_{i,r}^\pm$ (resp. by the $\phi_{i,r}^\pm$) for all $i \in I$, $r \in \mathbb{Z}$. Similarly, let $U^\pm$ (resp. U^0) be the subalgebra of $U_q(\mathfrak{g})$ generated by the $x_i^\pm$ (resp. by the $k_i^{\pm 1}$) for all $i \in I$.

Proposition 1.3. *(a)* $U_q(\mathfrak{g}) = U^-.U^0.U^+$.
(b) $U_q(\hat{\mathfrak{g}}) = \hat{U}^-.\hat{U}^0.\hat{U}^+$. $\square$

See [5] or [10] for details.

A representation W of $U_q(\mathfrak{g})$ is said to be of type 1 if it is the direct sum of its weight spaces

$$W_\lambda = \{w \in W \mid k_i.w = q_i^{\lambda(i)} w\}, \qquad (\lambda \in P).$$

If $W_\lambda \neq 0$, then λ is a weight of W. A vector $w \in W_\lambda$ is a highest weight vector if $x_i^+.w = 0$ for all $i \in I$, and W is a highest weight representation with highest weight λ if $W = U_q(\mathfrak{g}).w$ for some highest weight vector $w \in W_\lambda$.

It is known (see [5] or [10], for example) that every finite–dimensional irreducible representation of $U_q(\mathfrak{g})$ of type 1 is highest weight. Moreover, assigning to such a representation its highest weight defines a bijection between the set of isomorphism

classes of finite–dimensional irreducible type 1 representations of $U_q(\mathfrak{g})$ and P^+; the irreducible type 1 representation of $U_q(\mathfrak{g})$ of highest weight $\lambda \in P^+$ is denoted by $V(\lambda)$. Finally, every finite–dimensional representation W of $U_q(\mathfrak{g})$ is completely reducible: if W is of type 1, then

$$W \simeq \bigoplus_{\lambda \in P^+} V(\lambda)^{\oplus m_\lambda(W)}$$

for some uniquely determined multiplicities $m_\lambda(W) \in \mathbb{N}$. It is convenient to introduce the following notation: for $\mu \in P^+$, let

$$W_\mu^+ = \{w \in W_\mu : x_{i,0}^+.v = 0 \ \text{ for all } i \in I\}.$$

Then, $m_\mu(W) = \mathrm{d}im(W_\mu^+)$.

A representation V of $U_q(\hat{\mathfrak{g}})$ is of type 1 if $c^{1/2}$ acts as the identity on V, and if V is of type 1 as a representation of $U_q(\mathfrak{g})$. A vector $v \in V$ is a highest weight vector if

$$x_{i,r}^+.v = 0, \quad \phi_{i,r}^\pm.v = \Phi_{i,r}^\pm v, \quad c^{1/2}.v = v,$$

for some complex numbers $\Phi_{i,r}^\pm$. A type 1 representation V is a highest weight representation if $V = U_q(\hat{\mathfrak{g}}).v$, for some highest weight vector v, and the pair of $(I \times \mathbb{Z})$-tuples $(\Phi_{i,r}^\pm)_{i\in I, r\in\mathbb{Z}}$ is its highest weight. Note that $\Phi_{i,r}^+ = 0$ (resp. $\Phi_{i,r}^- = 0$) if $r < 0$ (resp. if $r > 0$), and that $\Phi_{i,0}^+\Phi_{i,0}^- = 1$. (In [5], highest weight representations of $U_q(\hat{\mathfrak{g}})$ are called 'pseudo-highest weight'.) Lowest weight representations are defined similarly.

If $\lambda \in P^+$, let $\mathcal{P}^\lambda$ be the set of all I-tuples $(P_i)_{i\in I}$ of polynomials $P_i \in \mathbb{C}[u]$, with constant term 1, such that $deg(P_i) = \lambda(i)$ for all $i \in I$. Set $\mathcal{P} = \cup_{\lambda\in P^+}\mathcal{P}^\lambda$.

Theorem 1.4. *(a) Every finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ can be obtained from a type 1 representation by twisting with an automorphism of $U_q(\hat{\mathfrak{g}})$.*

(b) Every finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ of type 1 is both highest and lowest weight.

(c) Let V be a finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ of type 1 and highest weight $(\Phi_{i,r}^\pm)_{i\in I, r\in\mathbb{Z}}$. Then, there exists $\mathbf{P} = (P_i)_{i\in I} \in \mathcal{P}$ such that

$$\sum_{r=0}^{\infty} \Phi_{i,r}^+ u^r = q_i^{deg(P_i)} \frac{P_i(q_i^{-2}u)}{P_i(u)} = \sum_{r=0}^{\infty} \Phi_{i,r}^- u^{-r},$$

in the sense that the left- and right-hand terms are the Laurent expansions of the middle term about 0 and ∞, respectively. Assigning to V the I-tuple $\mathbf{P}$ defines a bijection between the set of isomorphism classes of finite-dimensional irreducible representations of $U_q(\hat{\mathfrak{g}})$ of type 1 and $\mathcal{P}$. We denote by $V(\mathbf{P})$ the irreducible representation associated to $\mathbf{P}$.

(d) Let $\mathbf{P}$, $\mathbf{Q} \in \mathcal{P}$ be as above, and let $v_{\mathbf{P}}$ and $v_{\mathbf{Q}}$ be highest weight vectors of $V(\mathbf{P})$ and $V(\mathbf{Q})$, respectively. Then, in $V(\mathbf{P})\otimes V(\mathbf{Q})$,

$$x_{i,r}^+.(v_{\mathbf{P}}\otimes v_{\mathbf{Q}}) = 0, \quad \phi_{i,r}^\pm.(v_{\mathbf{P}}\otimes v_{\mathbf{Q}}) = \Psi_{i,r}^\pm(v_{\mathbf{P}}\otimes v_{\mathbf{Q}}),$$

where the complex numbers $\Psi_{i,r}^\pm$ are related to the polynomials P_iQ_i as the $\Phi_{i,r}^\pm$ are related to the P_i in part (c). In particular, if $\mathbf{P}\otimes\mathbf{Q}$ denotes the I-tuple (P_iQ_i)

then $V(\mathbf{P}\otimes\mathbf{Q})$ is isomorphic to a quotient of the subrepresentation of $V(\mathbf{P})\otimes V(\mathbf{Q})$ generated by $v_{\mathbf{P}}\otimes v_{\mathbf{Q}}$.

(e) If $\mathbf{P}=(P_i)_{i\in I}\in\mathcal{P}$, $a\in\mathbb{C}^\times$, and if $\tau_a^*(V(\mathbf{P}))$ denotes the pull-back of $V(\mathbf{P})$ by the automorphism τ_a, we have

$$\tau_a^*(V(\mathbf{P}))\cong V(\mathbf{P}^a)$$

as representations of $U_q(\hat{\mathfrak{g}})$, where $\mathbf{P}^a=(P_i^a)_{i\in I}$ and

$$P_i^a(u)=P_i(au). \qquad \square$$

See [5] and [7] for further details. If the highest weight $(\Phi_{i,r}^{\pm})_{i\in I,r\in\mathbb{Z}}$ of V is given by an I-tuple $\mathbf{P}$ as in part (c), we shall often abuse notation by saying that V has highest weight $\mathbf{P}$.

If $a\in\mathbb{C}^\times$, $i\in I$, we denote the irreducible representation of $U_q(\hat{\mathfrak{g}})$ with defining polynomials

$$P_j=\begin{cases}1 & \text{if } j\neq i,\\ 1-a^{-1}u & \text{if } j=i\end{cases}$$

by $V(\lambda_i,a)$, and denote the highest (resp. lowest) weight vector in this representation by v_{λ_i} (resp. $v_{-\lambda_i}$).

For $i\in I$, the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by the elements $x_{i,r}^{\pm}$ ($r\in\mathbb{Z}$), $k_i^{\pm1}$ ($i\in I$), and $h_{i,r}$ ($i\in I$, $r\in\mathbb{Z}\backslash\{0\}$) is isomorphic to $U_{q_i}(\hat{sl}_2)$; we denote this subalgebra by $U_q(\hat{\mathfrak{g}}_{(i)})$. The subalgebra $U_q(\mathfrak{g}_{(i)})$ is defined similarly. Let $\mu_{(i)}$ be the restriction of μ to $\{i\}$. The following lemma was proved in [6].

Lemma 1.5. *Let M be any highest weight representation of $U_q(\hat{\mathfrak{g}})$ with highest weight P and highest weight vector m.*
(i) For $i=1,2$, $M_{(i)}=U_q(\hat{\mathfrak{g}}_{(i)}).m$ is a highest weight representation of $U_q(\hat{\mathfrak{g}}_{(i)})$ with highest weight P_i and

$$m_\mu(M)=m_{\mu_{(i)}}(M_{(i)}).$$

(ii) Let N be another highest weight representation of $U_q(\hat{\mathfrak{g}})$ with highest weight $\mathbf{Q}$ and assume that λ is the highest weight of $M\otimes N$ (i.e. $\lambda(i)=deg(P_i)+deg(Q_i)$ for $i=1,2$). Then, for $i=1,2$ and $r\in\mathbb{Z}_+$, we have

$$m_{\lambda-r\alpha_i}(M\otimes N)=m_{\lambda_{(i)}-r\alpha_i}(M_{(i)}\otimes N_{(i)}). \qquad \square$$

2 Minimal affinizations

Following [2], we say that a finite-dimensional irreducible representation V of $U_q(\hat{\mathfrak{g}})$ is an affinization of $\lambda\in P^+$ if $V\cong V(\mathbf{P})$ as a representation of $U_q(\hat{\mathfrak{g}})$, for some $\mathbf{P}\in\mathcal{P}^\lambda$. Two affinizations of λ are equivalent if they are isomorphic as representations of $U_q(\mathfrak{g})$; we denote by $[V]$ the equivalence class of V. Let $\mathcal{Q}^\lambda$ be the set of equivalence classes of affinizations of λ.

The following result is proved in [2].

Proposition 2.1. *If $\lambda \in P^+$ and $[V]$, $[W] \in \mathcal{Q}^\lambda$, we write $[V] \preceq [W]$ iff, for all $\mu \in P^+$, either,*

(i) $m_\mu(V) \leq m_\mu(W)$, or

(ii) there exists $\nu > \mu$ with $m_\nu(V) < m_\nu(W)$.

Then, $\preceq$ is a partial order on $\mathcal{Q}^\lambda$. $\square$

An affinization V of λ is minimal if $[V]$ is a minimal element of $\mathcal{Q}^\lambda$ for the partial order $\preceq$, i.e. if $[W] \in \mathcal{Q}^\lambda$ and $[W] \preceq [V]$ implies that $[V] = [W]$. It is proved in [2] that $\mathcal{Q}^\lambda$ is a finite set, so minimal affinizations certainly exist.

A necessary condition for minimality was obtained in [2]. To state this result, we recall that the set of complex numbers $\{aq^{-r+1}, aq^{-r+3}, \ldots, aq^{r-1}\}$ is called the q–segment of length $r \in \mathbb{N}$ and centre $a \in \mathbb{C}^\times$.

Proposition 2.2. *Let $\lambda \in P^+$, let $\mathbf{P} = (P_i)_{i\in I} \in \mathcal{P}^\lambda$, and assume that $V(\mathbf{P})$ is a minimal affinization of λ. Then, for all $i \in I$, the roots of P_i form a q_i-segment of length $\lambda(i)$.* $\square$

Note that it follows from 1.4(e) and 2.2 that, if $i \in I$ and $r \in \mathbb{N}$, the weight $r\lambda_i$ has a unique affinization, up to equivalence.

For the rest of this paper we assume that $\mathfrak{g}$ is of type B_2. In this case, the defining polynomials of the minimal affinizations were determined in [2]:

Theorem 2.3. *Let $\lambda \in P^+$ and $\mathbf{P} \in \mathcal{P}^\lambda$. Then, $V(\mathbf{P})$ is a minimal affinization of λ iff the following conditions are satisfied:*
(a) for each $i = 1, 2$, either $P_i = 1$ or the roots of P_i form a q_i–segment of length $\lambda(i)$ and centre a_i (say);
(b) if $P_1 \neq 1$ and $P_2 \neq 1$, then

$$\frac{a_1}{a_2} = q^{2\lambda(1)+\lambda(2)+1} \quad \text{or} \quad q^{-(2\lambda(1)+\lambda(2)+3)}.$$

Any two minimal affinizations of λ are equivalent. Finally, if $V(\mathbf{P})$ is a minimal affinization of λ and $r \in \mathbb{Z}_+\backslash\{0\}$, we have

$$m_{\lambda - r\alpha_1}(V(\mathbf{P})) = m_{\lambda - r\alpha_2}(V(\mathbf{P})) = m_{\lambda-\alpha_1-\alpha_2}(V(\mathbf{P})) = 0. \quad \square$$

Our concern in this paper is the structure of a minimal affinization $V(\mathbf{P})$ as a representation of $U_q(\mathfrak{g})$. Our main result is:

Theorem 2.4. *Let $\lambda \in P^+$ and let $V(\mathbf{P})$ be a minimal affinization of λ. Then, as a representation of $U_q(\mathfrak{g})$,*

$$V(\mathbf{P}) \cong \bigoplus_{r=0}^{\mathrm{int}(\frac{1}{2}\lambda(2))} V(\lambda - 2r\lambda_2).$$

Here, for any real number b, $\mathrm{int}(b)$ is the greatest integer less than or equal to b.

The proof of Theorem 2.4 is by induction on $\lambda(2)$. The first part of the following proposition begins the induction.

Proposition 2.5.
(a) For any $r \in \mathbb{N}$, the minimal affinization of $r\lambda_1$ is irreducible as a $U_q(\mathfrak{g})$–module.
(b) The minimal affinization of λ_2 is irreducible as a representation of $U_q(\mathfrak{g})$.

Proof. (a) Let $\mathbf{P} \in \mathcal{P}^{r\lambda_1}$ be such that $V(\mathbf{P})$ is a minimal affinization of $r\lambda_1$. The element $x_0^+.v_{\mathbf{P}}$ has weight $r_1\lambda_1 - \alpha_1 - 2\alpha_2$. This weight is Weyl group conjugate to $r_1\lambda_1 - \alpha_1 \in P^+$. Hence, if $m_\nu(V(\mathbf{P}) > 0$ and $x_0^+.v_{\mathbf{P}}$ has a non-zero component in a $U_q(\mathfrak{g})$-subrepresentation of $V(\mathbf{P})$ of highest weight ν, then $\nu = r_1\lambda_1$ or $r_1\lambda_1 - \alpha_1$. But, $m_{r_1\lambda_1-\alpha_1}(V(\mathbf{P})) = 0$ by 2.2 and so $x_0^+.v_{\mathbf{P}} \in U_q(\mathfrak{g}).v_{\mathbf{P}} \cong V(r_1\lambda_1)$. It follows that x_0^+ preserves $V(r\lambda_1)$. Working with a lowest weight vector of $V(\mathbf{P})$, one proves similarly that x_0^- preserves $U_q(\mathfrak{g}).v_{\mathbf{P}}$. Hence, $U_q(\mathfrak{g}).v_{\mathbf{P}}$ is a $U_q(\hat{\mathfrak{g}})$-subrepresentation of $V(\mathbf{P})$, hence is equal to $V(\mathbf{P})$, and so $V(\mathbf{P}) \cong V(r\lambda_1)$ as representations of $U_q(\mathfrak{g})$.

(b) This is obvious, since there is no $\mu \in P^+$ such that $\mu < \lambda_2$. $\square$

We conclude this section with the following result on the dual of $V(\lambda_2, a)$.

If V is any representation of $U_q(\hat{\mathfrak{g}})$, its left dual ${}^tV$ is the representation of $U_q(\hat{\mathfrak{g}})$ on the vector space dual of V given by

$$< a.f, v > \;=\; < f, S(a).v >, \qquad (a \in U_q(\hat{\mathfrak{g}}), v \in V, f \in {}^tV)$$

where S is the antipode of $U_q(\hat{\mathfrak{g}})$ and $<,>$ is the natural pairing between V and its dual. The right dual V^t is defined in the same way, replacing S by S^{-1}. Left and right duals of representations of $U_q(\mathfrak{g})$ are defined similarly. Clearly the (left or right) dual of an irreducible representation is again irreducible. In fact, it is well known that, for any $\lambda \in P^+$,

$${}^tV(\lambda) \cong V(\lambda)^t \cong V(-w_0\lambda),$$

where w_0 is the longest element of the Weyl group of $\mathfrak{g}$.

Lemma 2.6. *(i) For any $a \in \mathbb{C}^\times$,*

$$V(\lambda_2, a)^t \cong V(\lambda_2, aq^6), \qquad {}^tV(\lambda_2, a) \cong V(\lambda_2, aq^{-6}).$$

(ii) For any $a, b \in \mathbb{C}^\times$,

$$dim((V(\lambda_2, a) \otimes V(\lambda_2, b))_0^+) = 1.$$

Moreover, if $0 \neq v_0 \in (V(\lambda_2, a) \otimes V(\lambda_2, b))_0^+$ and $a/b \neq q^{\pm 6}$, then $x_0^\pm.v_0$ is a non–zero multiple of $v_{\mp\lambda_2} \otimes v_{\mp\lambda_2}$.

Proof. (i) Since, for any representation V of $U_q(\hat{\mathfrak{g}})$, the canonical isomorphism of vector spaces ${}^tV^t \to V$ is an isomorphism of representations, it suffices to prove the first formula. Since $V(\lambda_2)$ is a self-dual representation of $U_q(\mathfrak{g})$, we have *a priori* that $V(\lambda_2, a)^t \cong V(\lambda_2, b)$ for some $b \in \mathbb{C}^\times$.

Fix $v_{-\lambda_2} = x_2^- x_1^- x_2^-.v_{\lambda_2}$. Then, $v_{-\lambda_2}$ is a non–zero element of $V(\lambda_2, a)_{-\lambda_2}$ and, for weight reasons,

$$x_0^+.v_{\lambda_2} = Av_{-\lambda_2}$$

for some $A \in \mathbb{C}$. Let $0 \neq v_{\lambda_2}^t \in V(\lambda_2, a)_{\lambda_2}^t$. Then $< v_{\lambda_2}^t, w > = 0$ if $w \notin V(\lambda_2, a_2)_{-\lambda_2}$. Normalize $v_{\lambda_2}^t$ so that

$$< v_{\lambda_2}^t, v_{-\lambda_2} > = 1$$

and let $v^t_{-\lambda_2} = x_2^- x_1^- x_2^- .v^t_{\lambda_2}$. Again, for weight reasons, one has

$$x_0^+ .v^t_{\lambda_2} = Bv^t_{-\lambda_2}.$$

for some $B \in \mathbb{C}$. Moreover, from the formula for x_0^+ in 1.2, it is clear that

$$A = a^{-1}c, \quad B = b^{-1}c,$$

where $c \in \mathbb{C}^\times$ depends only on q, and not on a or b. Thus, $A/B = b/a$. But A/B may be computed as follows:

$$< x_0^+ .v^t_{\lambda_2}, v_{\lambda_2} > \;=\; < v^t_{\lambda_2}, S^{-1}(x_0^+).v_{\lambda_2} > = < v^t_{\lambda_2}, -k_0^{-1} x_0^+ .v_{\lambda_2} > .$$

Hence,

$$B < x_2^- x_1^- x_2^- .v^t_{\lambda_2}, v_{\lambda_2} > = -q^{-2} A.$$

Since

$$S(x_2^- x_1^- x_2^-).v_{\lambda_2} = q^4 x_2^- x_1^- x_2^- .v_2 = q^4 v_{-\lambda_2},$$

we find that $A/B = q^6$, and part (i) is proved.

(ii) Since $V(\lambda_2)$ is a self–dual representation of $U_q(\mathfrak{g})$, it follows from 2.4 that

$$\dim((V(\lambda_2, a) \otimes V(\lambda_2, b))_0^+) = 1.$$

If $x_0^\pm .v_0 = 0$, then $\mathbb{C}.v_0$ is a $U_q(\hat{\mathfrak{g}})$–subrepresentation of the tensor product, and hence by (i) we have $a/b = q^{\pm 6}$. $\square$

3 A first reduction

For the remainder of this paper, we assume that $\lambda \in P^+$, $\lambda(2) \geq 1$ and that 2.4 is known for $\lambda - \lambda_2$. We shall also assume that $\lambda(1) \geq 1$. The proof when $\lambda(1) = 0$ is similar and easier.

We also *fix for the rest of the paper* an element $\mathbf{P} = (P_i)_{i \in I} \in \mathcal{P}^\lambda$ such that the roots of P_i form a string with centre a_i and length $\lambda(i)$, $i = 1, 2$, and such that

$$\frac{a_1}{a_2} = q^{-(2\lambda(1) + \lambda(2) + 3)}.$$

Define an element $\mathbf{Q} \in \mathcal{P}^{\lambda - \lambda_2}$ by

$$Q_1 = P_1, \quad Q_2 = \prod_{i=1}^{\lambda(2)-1} (1 - a_2^{-1} q^{-(\lambda(2) - 2i - 1)} u).$$

By 2.3, $V(\mathbf{P})$ and $V(\mathbf{Q})$ are minimal affinizations of λ and $\lambda - \lambda_2$, respectively. In particular, 2.4 is known for $V(\mathbf{Q})$.

The following is the main result of this section:

Proposition 3.1. *Let $\lambda, \mu \in P^+$ and let $V(\mathbf{P})$ be a minimal affinization of λ as above. Then:*

(i) $m_\mu(V(\mathbf{P})) \le 1$ if μ is of the form $\lambda - r\theta - \alpha_2$ or $\lambda - r\theta - \alpha_1 - \alpha_2$ for some $r \in \mathbb{N}$.

(ii) $m_\mu(V(\mathbf{P})) \le 2$ if μ is of the form $\lambda - r\theta$ for some $r \in \mathbb{N}$.

(iii) $m_\mu(V(\mathbf{P})) = 0$ if μ is not of the form $\lambda - r\theta$, $\lambda - r\theta - \alpha_2$ or $\lambda - r\theta - \alpha_1 - \alpha_2$ for some $r \in \mathbb{N}$.

(iv) $m_{\lambda - r\theta}(V(\mathbf{P})) \ge 1$ for $0 \le r \le int(\frac{1}{2}\lambda(2))$.

We deduce this from the next two results.

Lemma 3.2. *For any $\lambda \in P^+$,*

$$V(\lambda)\otimes V(\lambda_2) \cong V(\lambda+\lambda_2)\oplus V(\lambda+\lambda_2-\alpha_2)\oplus V(\lambda+\lambda_2-\alpha_1-\alpha_2)\oplus V(\lambda+\lambda_2-\theta).$$

Proof. By 1.4(c), it suffices to prove the analogous classical result. We leave this to the reader. $\square$

Proposition 3.3. *Let $\lambda \in P^+$, $\mathbf{P} \in \mathcal{P}^\lambda$, $\mathbf{Q} \in \mathcal{P}^{\lambda-\lambda_2}$ be as defined above.*
(i) $V(\lambda_2, a_2q^{\lambda(2)-1})\otimes V(\mathbf{Q})$ is generated as a representation of $U_q(\hat{\mathfrak{g}})$ by the tensor product of the highest weight vectors. In particular, $V(\mathbf{P})$ is isomorphic to a quotient of $V(\lambda_2, a_2q^{\lambda(2)-1})\otimes V(\mathbf{Q})$.
(ii) Let $\mathbf{P}_{(1)} = (P_1, 1)$. Then, there exists a surjective homomorphism of representations of $U_q(\hat{\mathfrak{g}})$

$$\pi : V(\lambda_2, a_2q^{\lambda(2)-1})\otimes V(\lambda_2, a_2q^{\lambda(2)-3})\otimes\cdots\otimes V(\lambda_2, a_2q^{-\lambda(2)+1})\otimes V(\mathbf{P}_{(1)}) \to V(\mathbf{P})$$

such that $\pi(v_{\lambda_2}^{\otimes\lambda(2)}\otimes v_{\mathbf{P}_{(1)}}) = v_{\mathbf{P}}$.

We assume 3.3 for the moment and give the

Proof of 3.1. Parts (i), (ii) and (iii) are easy consequences of 2.4(ii), 3.2 and 3.3(i), since 2.4 is known for $V(\mathbf{Q})$.

To prove (iv), we can assume that $\lambda(2) \ge 2$, since otherwise there is nothing to prove. Notice that, by 2.5, we can (and do) choose elements $0 \ne w_s \in (V(\lambda_2, a_2q^{\lambda(2)-4s+3})\otimes V(\lambda_2, a_2q^{\lambda(2)-4s+1}))_0^+$ such that

$$x_0^-.w_s = v_{\lambda_2}\otimes v_{\lambda_2}.$$

For $1 \le r \le \mathrm{int}(\frac{1}{2}\lambda(2))$, consider the element $w = w_1\otimes w_2\otimes\cdots w_r\otimes v_{\lambda_2}^{\otimes\lambda(2)-2r}\otimes v_{\mathbf{P}_{(1)}} \in V(\lambda_2, a_2q^{\lambda(2)-1})\otimes V(\lambda_2, a_2q^{\lambda(2)-3})\otimes\cdots\otimes V(\lambda_2, a_2q^{-\lambda(2)+1})\otimes V(\mathbf{P}_{(1)})$. Clearly, $x_{i,0}^+.w = 0$ for $i = 1, 2$, and an easy computation shows that

$$(x_0^-)^r.w = q^{r(r-1)}[r]_{q^2}v_{\lambda_2}^{\otimes\lambda(2)}\otimes v_{\mathbf{P}_{(1)}}.$$

Hence, $\pi((x_0^-)^r.w) \ne 0$ and so $\pi(w)$ is a non–zero element of $V(\mathbf{P})_{\lambda-r\theta}^+$. This proves 3.1(iv). $\square$

Proof of 3.3. Assuming 3.3(i) we give the proof of 3.3(ii). The proof is by induction on $\lambda(2)$. The case $\lambda(2) = 1$ is just 3.3(i). So if $\lambda(2) > 1$, by the induction hypothesis applied to $\lambda - \lambda_2$, we have a surjective homomorphism of representations of $U_q(\hat{\mathfrak{g}})$

$$\pi' : V(\lambda_2, a_2q^{\lambda(2)-3})\otimes\cdots\otimes V(\lambda_2, a_2q^{-\lambda(2)+1})\otimes V(\mathbf{P}_{(1)}) \to V(\mathbf{Q})$$

Consider

$$\mathrm{id}\otimes\pi' : V(\lambda_2, a_2q^{\lambda(2)-1})\otimes\cdots\otimes V(\lambda_2, a_2q^{-\lambda(2)+1})\otimes V(\mathbf{P}_{(1)}) \to V(\lambda_2, a_2q^{\lambda(2)-1})\otimes V(\mathbf{Q}).$$

By 3.3(i), the right-hand side has $V(\mathbf{P})$ as a quotient and so we get the required surjective homomorphism

$$\pi : V(\lambda_2, a_2q^{\lambda(2)-1})\otimes\cdots\otimes V(\lambda_2, a_2q^{-\lambda(2)+1})\otimes V(\mathbf{P}_{(\mathbf{1})}) \to V(\mathbf{P}).$$

We now prove 3.3(i). Let $M = U_q(\hat{\mathfrak{g}}).(v_{\lambda_2}\otimes v_{\mathbf{Q}})$. We first show that it suffices to prove

$$m_\mu(M) = m_\mu(V(\lambda_2, a_2q^{\lambda(2)-1})\otimes V(\mathbf{Q})) \quad \text{for} \;\; \mu > \lambda - \theta. \tag{3}$$

To see this, assume that M is a proper subrepresentation of the tensor product and let N be the corresponding quotient. It follows from 3.2 and (3) that

$$m_\mu(N) = 0 \quad \text{unless} \;\; \mu \le \lambda - \theta.$$

On the other hand, dualizing the projection map

$$V(\lambda_2, a_2q^{\lambda(2)-1})\otimes V(\mathbf{Q}) \to N$$

we get a non–zero (hence injective) homomorphism of representations of $U_q(\hat{\mathfrak{g}})$

$$V(\mathbf{Q}) \to V(\lambda_2, a_2q^{\lambda(2)})^t\otimes N.$$

It follows that

$$m_{\lambda-\lambda_2}(V(\lambda_2)\otimes N) \ge 1,$$

and hence by 3.2 that $m_{\lambda-\theta}(N) > 0$.

Note that the preceding argument proves that, if M' is any $U_q(\hat{\mathfrak{g}})$–subrepresentation of $V(\lambda_2, a_2q^{\lambda(2)-1})\otimes V(\mathbf{Q})$ containing M, and if N' is the corresponding quotient of the tensor product, then $m_{\lambda-\theta}(N') > 0$. In particular, any irreducible quotient N' of N must have $m_{\lambda-\theta}(N') > 0$. Taking N' to be an affinization $V(\mathbf{R})$, say, of $\lambda-\theta$, we have a surjective map of $U_q(\hat{\mathfrak{g}})$–representations

$$V(\lambda_2, a_2q^{\lambda(2)})\otimes V(\mathbf{Q}) \to V(\mathbf{R}),$$

and hence, dualizing, an injective map

$$V(\mathbf{Q}) \to V(\lambda_2, a_2q^{\lambda(2)-1})^t\otimes V(\mathbf{R}) = V(\lambda_2, a_2q^{\lambda(2)+5})\otimes V(\mathbf{R}),$$

by 2.5. The highest weight vector in $V(\mathbf{Q})$ must map to (a non–zero multiple of) the tensor product of the highest weight vectors on the right-hand side. But this is impossible, since $a_2q^{\lambda(2)+5}$ is not a root of Q_2. Hence, $N = 0$ and part (i) is proved.

We now prove (3). The statement is obviously true for $\mu = \lambda$. For $\mu = \lambda-\alpha_1$, the statement follows from 3.2 and the fact that 3.4 is known for $V(\mathbf{Q})$. For

notice that, by 1.5, it suffices to prove the result for $U_q(\hat{sl}_2)$. But this follows from 4.9(a) of [3] since

$$V(\lambda_2, a_2q^{\lambda(2)-1})_{(2)}\otimes V(\mathbf{Q})_{(2)} \cong V(1, a_2q^{\lambda(2)-1})\otimes V(\lambda(2)-1, a_2q^{-1})$$

as representations of $U_q(\hat{\mathfrak{g}}_{(2)})$.

Finally, we must prove (3) for $\lambda-\alpha_1-\alpha_2$. For this, it obviously suffices to prove that

$$(V(\lambda_2, a_2q^{\lambda(2)-1})\otimes V(\mathbf{Q}))_{\lambda-\alpha_1-\alpha_2} \subseteq M.$$

The left-hand side is spanned by

$$\{x_1^-x_2^-.v_{\lambda_2}\otimes v_{\mathbf{Q}}, v_{\lambda_2}\otimes x_1^-x_2^-.v_{\mathbf{Q}}, v_{\lambda_2}\otimes x_2^-x_1^-.v_{\mathbf{Q}}, x_2^-.v_{\lambda_2}\otimes x_1^-.v_{\mathbf{Q}}\}.$$

Now, since $m_\lambda(M)$ and $m_{\lambda-\alpha_2}(M)$ are both strictly positive, M contains $x_2^-.v_{\lambda_2}\otimes v_{\mathbf{Q}}$ and $v_{\lambda_2}\otimes x_2^-.v_{\mathbf{Q}}$ (since M contains two linear combinations of these vectors which are not scalar multiples of each other). Also, M contains

$$v_{\lambda_2}\otimes x_1^-.v_{\mathbf{Q}} = x_1^-.(v_{\lambda_2}\otimes v_{\mathbf{Q}}).$$

It follows that M contains the three vectors

$$x_1^-.(x_2^-.v_{\lambda_2}\otimes v_{\mathbf{Q}}),\quad x_1^-.(v_{\lambda_2}\otimes x_2^-.v_{\mathbf{Q}}),\quad x_2^-.(v_{\lambda_2}\otimes x_1^-.v_{\mathbf{Q}}),$$

i.e. that M contains the three vectors

$$\begin{gathered} v_{\lambda_2}\otimes x_1^-x_2^-.v_{\mathbf{Q}}, \\ x_1^-x_2^-.v_{\lambda_2}\otimes v_{\mathbf{Q}} + q^{-2}x_2^-.v_{\lambda_2}\otimes x_1^-.v_{\mathbf{Q}}, \\ x_2^-.v_{\lambda_2}\otimes x_1^-.v_{\mathbf{Q}} + q^{-1}v_{\lambda_2}\otimes x_2^-x_1^-.v_{\mathbf{Q}}. \end{gathered} \tag{4}$$

Since these vectors are obviously linearly independent, it suffices to prove that

$$[x_2^+, x_0^+].(v_{\lambda_2}\otimes v_{\mathbf{Q}}) \tag{5}$$

is linearly independent of the vectors in (4).

To compute the vector in (5), we need the following formulas:

$$x_{1,1}^-.v_{\mathbf{Q}} = a_1^{-1}q^{2\lambda(1)-2}x_1^-.v_{\mathbf{Q}}, \tag{6}$$

$$x_{1,1}^-x_2^-.v_{\mathbf{Q}} = a_1^{-1}q^{2\lambda(1)-2}x_1^-x_2^-.v_{\mathbf{Q}}. \tag{7}$$

By the formula for the isomorphism f in 1.2,

$$[x_2^+, x_0^+] = c[x_2^-, x_{1,1}^-]_q(k_1k_2)^{-1},$$

where $c \in \mathbb{C}^\times$, and for any $x, y \in U_q(\hat{\mathfrak{g}})$, we define

$$[x, y]_q = qxy - q^{-1}yx$$

Using this, we find that

$$
\begin{aligned}
[x_2^+, x_0^+].(v_{\lambda_2} \otimes v_{\mathbf{Q}}) &= cq^{-(2\lambda(1)+\lambda(2))}(q^{-1}[x_2^-, x_{1,1}^-]_q.v_{\lambda_2} \otimes v_{\mathbf{Q}} + v_{\lambda_2} \otimes [x_2^-, x_{1,1}^-]_q.v_{\mathbf{Q}}) \\
&= cq^{-(2\lambda(1)+\lambda(2))}(-a_2^{-1}q^{-\lambda(2)+1}x_1^- x_2^-.v_{\lambda_2} \otimes v_{\mathbf{Q}} \\
&\qquad + v_{\lambda_2} \otimes (a_1^{-1}q^{2\lambda(1)-1}x_2^- x_1^-.v_{\mathbf{Q}} - a_2^{-1}q^{\lambda(2)-3}x_1^- x_2^-.v_{\mathbf{Q}})).
\end{aligned}
$$

An easy computation shows that this is linearly dependent on the vectors in (5) iff

$$
\frac{a_1}{a_2} = q^{2\lambda(1)+\lambda(2)+2},
$$

contradicting our assumption that $a_1/a_2 = q^{-(2\lambda(1)+\lambda(2)+2)}$.

The proof of (6) is easy since we know from 2.2 that $x_{1,1}^-.v_{\mathbf{Q}}$ must be a scalar multiple of $x_1^-.v_{\mathbf{Q}}$. As for (7), observe that by 2.2 again we know *a priori* that

$$
x_{1,1}^- x_2^-.v_{\mathbf{Q}} = Ax_1^- x_2^-.v_{\mathbf{Q}} + Bx_2^- x_1^-.v_{\mathbf{Q}}
$$

for some $A, B \in \mathbb{C}$. Applying x_1^+ and x_2^+ to both sides of gives the pair of equations

$$
\begin{aligned}
A[\lambda(2)-1]_q + B[\lambda(2)]_q &= [\lambda(2)-1]_q a_1^{-1} q^{2\lambda(1)-2}, \\
A[\lambda(1)+1]_{q^2} + B[\lambda(1)]_{q^2} &= q^{2\lambda(1)a_1^{-1}}[\lambda(1)]_{q^2} + a_2^{-1}q^{(2\lambda(1)+\lambda(2)+1)}.
\end{aligned}
$$

Using $a_1/a_2 = q^{-(2\lambda(1)+\lambda(2)+3)}$, we find that the unique solution is $A = a_1^{-1}q^{2\lambda(1)-2}$, $B = 0$. $\square$

4. Completion of the proof of Theorem 2.4

In view of 3.3, to complete the proof of 2.4, it suffices to establish

Proposition 4.1. *Let $\lambda \in P^+$ and let $V(\mathbf{P})$ be a minimal affinization of λ. Then:*
(i) $m_{\lambda - r\theta}(V(\mathbf{P})) = 1$ if $0 \le r \le int(\frac{1}{2}\lambda(2))$.
(ii) $m_\mu(V(\mathbf{P})) = 0$ if μ is of the form $\lambda - r\theta - \alpha_2$, for some $r \in \mathbb{N}$.
(iii) $m_\mu(V(\mathbf{P})) = 0$ if μ is of the form $\lambda - r\theta - \alpha_1 - \alpha_2$ for some $r \in \mathbb{N}$.

We need three lemmas.

Lemma 4.2. *Suppose that there exists $0 \ne v \in V(\mathbf{P})_\mu^+$ such that*

$$
x_{1,1}^+.v = x_{1,-1}^+.v = 0
$$

(resp. $x_{2,1}^+.v = x_{2,-1}^+.v = 0$). Assume that $m_{\mu+\alpha_i}(V(\mathbf{P})) = 0$ for $i = 1, 2$. Then, $\lambda = \mu$.

Proof. We prove, by induction on $k \in \mathbb{N}$, that

(8) $\qquad x^{+} \ldots v = 0, \text{ for all } i = 1, 2$

It is easy to see using the relations in 1.2 that the k_j and $h_{j,s}$ preserve the finite–dimensional space

$$V(\mathbf{P})_\mu^{++} = \{w \in V(\mathbf{P})_\mu : x_{i,k}^+.w = 0 \text{ forall } \ i \in I, k \in \mathbb{Z}\}.$$

It follows that there exists a $U_q(\hat{\mathfrak{g}})$–highest weight vector in $V(\mathbf{P})_\mu$, which is possible only if $\lambda = \mu$.

It is obvious that (8) holds when $k = 0$. We assume that it holds for k and prove it for $k+1$. Using (1), we find that

$$x_{j,0}^+ x_{i,\pm(k+1)}^+ \in U_q(\hat{\mathfrak{g}})x_{j,0}^+ + U_q(\hat{\mathfrak{g}})x_{j,\pm1}^+ + U_q(\hat{\mathfrak{g}})x_{i,\pm k}^+,$$

and hence by the induction hypotheses we see that $x_{i,\pm k+1}^+.v \in V(\mathbf{P})_{\mu+\alpha_i}^+$. Since $m_{\mu+\alpha_i}(V(\mathbf{P})) = 0$ by assumption, this forces $x_{i,\pm k+1}^+.v = 0$, establishing (8) for $k+1$. $\square$

Lemma 4.3. *Let $0 \neq v \in V(\mathbf{P})_\mu$ be such that $x_{1,s'}^+.v = 0$ if $0 \le s' < s$ or if $s < s' \le 0$. Then*
(i) $(x_{2,0}^+)^3 x_{1,s}^+.v = 0$,
(ii) $x_{1,0}^+ x_{2,0}^+ x_{1,s}^+.v = 0$,
(iii) $x_{1,0}^+(x_{2,0}^+)^2 x_{1,s}^+.v \in V(\mathbf{P})_{\mu+2\alpha_1+2\alpha_2}^+$.

Proof. Using relation (2) we find that

$$(x_{2,0}^+)^3 x_{1,\pm s}^+ \in U_q(\hat{\mathfrak{g}}).x_{2,0}^+. \tag{9}$$

Part (i) is now immediate.

For (ii), it suffices to notice that (1) and (2) together give

$$x_{1,0}^+ x_{2,0}^+ x_{1,\pm s}^+ \in U_q(\hat{\mathfrak{g}})x_{2,0}^+ + \sum_{0 \le s' < s} U_q(\hat{\mathfrak{g}})x_{1,\pm s'}^+ \tag{10}$$

if $s > 0$.

For (iii), we use the following consequences of (1) and (2):

$$(x_{1,0}^+)^2 x_{2,0}^+ \in U_q(\hat{\mathfrak{g}})x_{1,0}^+, \tag{11}$$

$$x_{2,0}^+ x_{1,0}^+ (x_{2,0}^+)^2 \in U_q(\hat{\mathfrak{g}})(x_{2,0}^+)^3 + U_q(\hat{\mathfrak{g}})x_{1,0}^+ + U_q(\hat{\mathfrak{g}})x_{1,0}^+ x_{2,0}^+. \tag{12}$$

The result now follows from parts (i) and (ii). $\square$

Lemma 4.4. *Let $\mu \in P^+$ be such that*

$$m_{\mu+\eta}(V(\mathbf{P})) = 0 \quad \textit{if} \quad \eta \neq s\theta, s \in \mathbb{Z}_+. \tag{13}$$

Then, $(x_{2,0}^+)^2 x_{1,\pm1}^+$ maps $V(\mathbf{P})_\mu^+$ to $V(\mathbf{P})_{\mu+\theta}^+$. Further, if $v \in V(\mathbf{P})_\mu^+$ is such that $x_{1,\pm1}^+.v \neq 0$, then

$$(x_{2,0}^+)^2 x_{1,\pm1}^+ \neq 0.$$

Proof. It is clear for weight reasons that $(x_{2,0}^+)^2 x_{1,\pm1}^+$ maps $V(\mathbf{P})_\mu^+$ to $V(\mathbf{P})_{\mu+\theta}$. Thus, it suffices to prove that

$$x^+ (x^+)^2 x^+ .v = 0 \ \text{ if } \ v \in V(\mathbf{P})^+$$

For $i = 2$, this is just 4.3(i). For $i = 1$, the result is obvious from 4.3(iii) and (13).

Now suppose that $(x_{2,0}^+)^2 x_{1,1}^+.v = 0$. By 4.3(ii), $x_{1,0}^+ x_{2,0}^+ x_{1,1}^+.v = 0$ as well and (13) now forces

$$x_{2,0}^+ x_{1,1}^+.v = 0.$$

Now, (2) gives $x_{1,0}^+ x_{1,1}^+.v = 0$ and so by a final application (13), we get

$$x_{1,1}^+.v = 0.$$

One proves similarly that $(x_{2,0}^+)^2 x_{1,-1}^+.v = 0$ implies that $x_{1,-1}^+.v = 0$, and the proof of 4.4 is now complete. □

We are now in a position to give the

Proof of 4.1. All three parts are proved by induction on r. If $r = 0$, the result follows from 2.2. We assume that (i), (ii) and (iii) hold for r and prove them for $r + 1$.

(i) Suppose that $m_{\lambda-(r+1)\theta}(V(\mathbf{P})) > 1$. Then, by 4.4, there exists $0 \neq v_0 \in V(\mathbf{P})^+_{\lambda-(r+1)\theta}$ such that $x_{1,1}^+.v = 0$.

Suppose now that $x_{1,-1}^+.v \neq 0$. For $s = 0, 1, \ldots, r + 1$, define $v_s \in V(\mathbf{P})$ by

$$v_s = ((x_{2,0}^+)^2 x_{1,-1}^+)^s.v.$$

We claim that the v_s have the following properties:

(i)$_s$ $0 \neq v_s \in V(\mathbf{P})^+_{\lambda-(r+1-s)\theta}$ for all $0 \leq s \leq r + 1$;

(ii)$_s$ $x_{i,k}^+.v_s = 0$ for $i = 1, 2$, $k \geq 0$.

Note that (i)$_0$ holds by assumption and (ii)$_0$ by the choice of v_0. Assuming that these properties hold for s we now prove that they hold for $s+1$. Lemma 4.2 implies that $x_{1,-1}^+.v_s \neq 0$ if $0 \leq s \leq r$ and 4.4 now shows that $v_{s+1} \neq 0$. To prove that (ii)$_{s+1}$ holds, observe that, by the proof of 4.2, it suffices to prove that $x_{1,1}^+.v = 0$. Using (2) we find that

$$x_{1,1}^+ (x_{2,0}^+)^2 x_{1,-1}^+ \in U_q(\hat{\mathfrak{g}}) x_{1,1}^+ x_{2,0}^+ x_{1,-1}^+ + U_q(\hat{\mathfrak{g}}) x_{2,1}^+ x_{2,0}^+ x_{1,-1}^+ + U_q(\hat{\mathfrak{g}}) x_{1,0}^+ x_{2,0}^+ x_{1,-1}^+.$$

The third term kills v_s by 4.3(ii); on the other hand, using (1), we find that the first two terms are contained in

$$\sum_{i=1}^{2} (U_q(\hat{\mathfrak{g}}) x_{i,0}^+ + U_q(\hat{\mathfrak{g}}) x_{i,1}^+),$$

and hence kill v_s as well. This proves the claim.

Note that $v_{r+1} = A v_{\mathbf{P}}$, for some $A \in \mathbb{C}^\times$. Since $\dim(V(\mathbf{P})_{\lambda-\alpha_2}) = 1$, it follows that

$$x_{2,0}^+ x_{1,-1}^+.v_r = B x_{2,0}^-.v_{\mathbf{P}},$$

for some $B \in \mathbb{C}^\times$. Applying $x_{2,1}^+$ to both sides of this equation, and using (2) and (ii)$'_r$, we get

$$0 = B \phi_{2,1}^+ v_{\mathbf{P}}.$$

By 1.4(c), this is impossible, since $\lambda(2) > 0$. This completes the proof of 4.1(i).

(ii) Suppose that $m_{\lambda-(r+1)\theta-\alpha_2}(V(\mathbf{P})) > 0$. The induction hypotheses on r implies that

$$V(\mathbf{P})_{\lambda-r\theta-\eta} = 0 \ \text{ if } \ \eta = \alpha_2,\ 2\alpha_2,\ 3\alpha_2,\ \text{or}\ \alpha_2-\alpha_1. \tag{14}$$

Let $0 \neq v \in V(\mathbf{P})^+_{\lambda-(r+1)\theta-\alpha_2}$. We shall prove that v is actually $U_q(\hat{\mathfrak{g}})$–highest weight, which is obviously impossible. We first prove, by induction on k, that $x^+_{1,k}.v = 0$. By (14), it suffices to prove that $x^+_{1,k+1}.v \in V(\mathbf{P})^+_{\lambda-r\theta-3\alpha_2}$. Since $x^+_{1,0}x^+_{1,k+1} \in \sum_{0\le s<k+1} U_q(\hat{\mathfrak{g}})x^+_{1,s}$, we see that

$$x^+_{1,0}.x^+_{1,k+1}.v = 0.$$

To prove that $x^+_{2,0}x^+_{1,k+1}.v = 0$, define $v' = (x^+_{2,0})^2x^+_{1,k+1}.v$ and $v'' = x^+_{1,0}.v'$. Then, by (14),

$$\begin{aligned}
4.3(\text{iii}) &\implies v'' \in V(\mathbf{P})^+_{\lambda-r\theta-\alpha_2+\alpha_1} \implies v'' = 0,\\
4.3(\text{i}) &\implies v' \in V(\mathbf{P})^+_{\lambda-r\theta-\alpha_2} \implies v' = 0,\\
4.3(\text{ii}) &\implies x^+_{2,0}x^+_{1,k+1}.v \in V(\mathbf{P})^+_{\lambda-r\theta-3\alpha_2} \implies x^+_{2,0}x^+_{1,k+1}.v = 0.
\end{aligned}$$

To prove that $x^+_{2,k}.v = 0$, we again proceed by induction on k. We assume that $k \ge 0$; the proof for $k \le 0$ is similar.

Using (1) and the fact that $x^+_{1,k}.v = 0$ for all k, we see that

$$\begin{aligned}
&x^+_{1,r}x^+_{2,k+1}.v = 0, \quad \text{for } r = -1, 0, \text{ and } 1,\\
&x^+_{2,0}x^+_{2,k+1}.v = 0.
\end{aligned}$$

But now 4.2 implies that $x^+_{2,k+1}.v = 0$ (since $\lambda-(r+1)\theta \neq \lambda$). This completes the proof of 4.1(ii).

(iii) Let $v \in V(\mathbf{P})_{\lambda-(r+1)\theta-\alpha_1-\alpha_2}$. Since

$$m_{\lambda-(r+1)\theta-\alpha_i}(V(\mathbf{P})) = 0, \quad \text{for} \quad i = 1, 2,$$

and since $\lambda \neq \lambda-(r+1)\theta-\alpha_1-\alpha_2$, it suffices by 4.2 to prove that $x^+_{2,\pm1}.v = 0$. To do this, note that by 3.1, it is enough to prove that $x^+_{2,\pm1}.v \in V(\mathbf{P})^+_{\lambda-(r+1)\theta-\alpha_1}$. Clearly, by (1), $x^+_{2,0}x^+_{2,\pm1}.v = 0$.

To prove that $x^+_{1,0}x^+_{2,\pm1}.v = 0$, it suffices by 4.2 to prove that

$$x^+_{1,0}.x^+_{2,\pm1}.v \in V(\mathbf{P})^+_{\lambda-(r+1)\theta}, \tag{15}$$

$$x^+_{1,s}.x^+_{1,0}x^+_{2,\pm1}.v = 0 \ \text{ for } \ s = \pm1. \tag{16}$$

The fact that $(x^+_{1,0})^2x^+_{2,\pm1}.v = 0$ is clear from (2). By using (1) and (2), it is easy to see that $x^+_{2,0}x^+_{1,0}x^+_{2,\pm1}.v \in V(\mathbf{P})_{\lambda-r\theta-\alpha_1-\alpha_2}$, and hence must be zero by 3.1. This proves (15).

To prove (16), one checks first, using (1) and (2), that

It follows that $(x_{2,0}^+)^2 x_{1,s}^+ x_{1,0}^+ x_{2,\pm 1}^+.v = 0$ for $s = 0, 1$. Lemma 4.4 now implies that in fact

$$x_{1,s}^+ x_{1,0}^+ x_{2,\pm 1}^+.v = 0 \text{ for } s = \pm 1.$$

This completes the proof of 4.1(iii). $\square$

The proof of Theorem 2.4 is now complete.

References

1. Beck, J., Braid group action and quantum affine algebras, preprint, MIT, 1993.
2. Chari, V., Minimal affinizations of representations of quantum groups: the rank 2 case, preprint, 1994.
3. Chari, V. and Pressley, A. N., Quantum affine algebras, Commun. Math. Phys. **142** (1991), 261–83.
4. Chari, V. and Pressley, A. N., Small representations of quantum affine algebras, Lett. Math. Phys. **30** (1994), 131–45.
5. Chari, V. and Pressley, A. N., *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994.
6. Chari, V. and Pressley A. N., Minimal affinizations of representations of quantum groups: the simlpy-laced case, preprint, 1994.
7. Chari, V. and Pressley, A. N., Quantum affine algebras and their representations, preprint, 1994.
8. Delius, G.W. and Zhang, Y.-Z., Finite-dimensional representations of quantum affine algebras, preprint, 1994.
9. Drinfel'd, V. G., A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl. **36** (1988), 212–6.
10. Lusztig, G., *Introduction to Quantum Groups*, Progress in Mathematics 110, Birkhäuser, Boston, 1993.